

Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate

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Abstract

In this paper, we would like to derive a quantitative uniqueness estimate, the three-region inequality, for the second order elliptic equation with jump discontinuous coefficients. The derivation of the inequality relies on the Carleman estimate proved in our previous work [5]. We then apply the three-region inequality to study the size estimate problem with one boundary measurement.

1 Introduction

In this work we aim to study the size estimate problem with one measurement when the background conductivity has jump interfaces. A typical application of this study is to estimate the size of a cancerous tumor inside an organ by the electric impedance tomography (EIT). In this case, considering discontinuous medium is typical, for instance, the conductivities of heart, liver, intestines are 0.70 (S/m), 0.10 (S/m), 0.03 (S/m), respectively. Previous works on this problem assumed that the conductivity of the studied body is Lipschitz continuous, see, for example, [3, 4]. The first result on the size estimate problem with a discontinuous background conductivity was given in [17], where only the two dimensional case was considered. In this paper, we will study the problem in dimension $n \geq 2$.

The main ingredients of our method are quantitative uniqueness estimates for

$$\operatorname{div}(A\nabla u) = 0 \quad \Omega \subset \mathbb{R}^n. \quad (1.1)$$

Those estimates are well-known when A is Lipschitz continuous. The derivation of the estimates is based on the Carleman estimate or the frequency function method. For $n = 2$ and $A \in L^\infty$, quantitative uniqueness estimates are obtained via the connection

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between (1.1) and quasiregular mappings. This is the method used in [17]. For $n \geq 3$, the connection with quasiregular mappings is not true. Hence we return to the old method – the Carleman estimate, to derive quantitative uniqueness estimates when A is discontinuous. Precisely, when A has a $C^{1,1}$ interface and is Lipschitz away from the interface, a Carleman estimate was obtained in [5] (see [10, 11, 12] for related results). Here we will derive three-region inequalities using this Carleman estimate. The three-region inequality provides us a way to propagate "smallness" across the interface (see also [11] for similar estimates). Relying on the three-region inequality, we then derive bounds of the size of an inclusion with one boundary measurement. For other results on the size estimate, we mention [1] for the isotropic elasticity, [14, 15, 16] for the isotropic/anisotropic thin plate, [7, 6] for the shallow shell.

2 The Carleman estimate

In this section, we would like to describe the Carleman estimate derived in [5]. We first denote $H_{\pm} = \chi_{\mathbb{R}_{\pm}^n}$ where $\mathbb{R}_{\pm}^n = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \gtrless 0\}$ and $\chi_{\mathbb{R}_{\pm}^n}$ is the characteristic function of \mathbb{R}_{\pm}^n . Let $u_{\pm} \in C^{\infty}(\mathbb{R}^n)$ and define

$$u = H_+ u_+ + H_- u_- = \sum_{\pm} H_{\pm} u_{\pm},$$

hereafter, $\sum_{\pm} a_{\pm} = a_+ + a_-$, and

$$\mathcal{L}(x, y, \partial)u := \sum_{\pm} H_{\pm} \operatorname{div}_{x,y}(A_{\pm}(x, y) \nabla_{x,y} u_{\pm}), \quad (2.1)$$

where

$$A_{\pm}(x, y) = \{a_{ij}^{\pm}(x, y)\}_{i,j=1}^n, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R} \quad (2.2)$$

is a Lipschitz symmetric matrix-valued function satisfying, for given constants $\lambda_0 \in (0, 1]$, $M_0 > 0$,

$$\lambda_0 |z|^2 \leq A_{\pm}(x, y) z \cdot z \leq \lambda_0^{-1} |z|^2, \quad \forall (x, y) \in \mathbb{R}^n, \forall z \in \mathbb{R}^n \quad (2.3)$$

and

$$|A_{\pm}(x', y') - A_{\pm}(x, y)| \leq M_0(|x' - x| + |y' - y|). \quad (2.4)$$

We write

$$h_0(x) := u_+(x, 0) - u_-(x, 0), \quad \forall x \in \mathbb{R}^{n-1}, \quad (2.5)$$

$$h_1(x) := A_+(x, 0) \nabla_{x,y} u_+(x, 0) \cdot \nu - A_-(x, 0) \nabla_{x,y} u_-(x, 0) \cdot \nu, \quad \forall x \in \mathbb{R}^{n-1}, \quad (2.6)$$

where $\nu = -e_n$.

For a function $h \in L^2(\mathbb{R}^n)$, we define

$$\hat{h}(\xi, y) = \int_{\mathbb{R}^{n-1}} h(x, y) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^{n-1}.$$

As usual $H^{1/2}(\mathbb{R}^{n-1})$ denotes the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying

$$\int_{\mathbb{R}^{n-1}} |\xi| |\hat{f}(\xi)|^2 d\xi < \infty,$$

with the norm

$$\|f\|_{H^{1/2}(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} (1 + |\xi|^2)^{1/2} |\hat{f}(\xi)|^2 d\xi. \quad (2.7)$$

Moreover we define

$$[f]_{1/2, \mathbb{R}^{n-1}} = \left[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^n} dy dx \right]^{1/2},$$

and recall that there is a positive constant C , depending only on n , such that

$$C^{-1} \int_{\mathbb{R}^{n-1}} |\xi| |\hat{f}(\xi)|^2 d\xi \leq [f]_{1/2, \mathbb{R}^{n-1}}^2 \leq C \int_{\mathbb{R}^{n-1}} |\xi| |\hat{f}(\xi)|^2 d\xi,$$

so that the norm (2.7) is equivalent to the norm $\|f\|_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2, \mathbb{R}^{n-1}}$. From now on, we use the letters C, C_0, C_1, \dots to denote constants (depending on λ_0, M_0, n). The value of the constants may change from line to line, but it is always greater than 1. We will denote by $B_r(x)$ the $(n-1)$ -ball centered at $x \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x = 0$ we denote $B_r = B_r(0)$.

thm8.2

Theorem 2.1 *Let u and $A_{\pm}(x, y)$ satisfy (2.1)-(2.6). There exist $L, \beta, \delta_0, r_0, \tau_0$ positive constants, with $r_0 \leq 1$, depending on λ_0, M_0, n , such that if $\alpha_+ > L\alpha_-$, $\delta \leq \delta_0$ and $\tau \geq \tau_0$, then*

$$\begin{aligned} & \sum_{\pm} \sum_{|k|=0}^2 \tau^{3-2|k|} \int_{\mathbb{R}_{\pm}^n} |D^k u_{\pm}|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy + \sum_{\pm} \sum_{|k|=0}^1 \tau^{3-2|k|} \int_{\mathbb{R}^{n-1}} |D^k u_{\pm}(x, 0)|^2 e^{2\phi_{\delta}(x, 0)} dx \\ & + \sum_{\pm} \tau^2 [e^{\tau\phi_{\delta}(\cdot, 0)} u_{\pm}(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2 + \sum_{\pm} [D(e^{\tau\phi_{\delta, \pm}} u_{\pm})(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2 \\ & \leq C \left(\sum_{\pm} \int_{\mathbb{R}_{\pm}^n} |\mathcal{L}(x, y, \partial)(u_{\pm})|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy + [e^{\tau\phi_{\delta}(\cdot, 0)} h_1]_{1/2, \mathbb{R}^{n-1}}^2 \right. \\ & \quad \left. + [\nabla_x(e^{\tau\phi_{\delta}} h_0)(\cdot, 0)]_{1/2, \mathbb{R}^{n-1}}^2 + \tau^3 \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau\phi_{\delta}(x, 0)} dx + \tau \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau\phi_{\delta}(x, 0)} dx \right). \end{aligned} \quad (2.8)$$

where $u = H_+ u_+ + H_- u_-$, $u_{\pm} \in C^\infty(\mathbb{R}^n)$ and $\text{supp } u \subset B_{\delta/2} \times [-\delta r_0, \delta r_0]$, and $\phi_{\delta, \pm}(x, y)$ is given by

$$\phi_{\delta, \pm}(x, y) = \begin{cases} \frac{\alpha_+ y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}, & y \geq 0, \\ \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}, & y < 0, \end{cases} \quad (2.9)$$

and $\phi_{\delta}(x, 0) = \phi_{\delta, +}(x, 0) = \phi_{\delta, -}(x, 0)$.

Remark 2.2 It is clear that (2.8) remains valid if can add lower order terms $\sum_{\pm} H_{\pm} (W \cdot \nabla_{x,y} u_{\pm} + V u_{\pm})$, where W, V are bounded functions, to the operator \mathcal{L} . That is, one can substitute

$$\mathcal{L}(x, y, \partial)u = \sum_{\pm} H_{\pm} \operatorname{div}_{x,y} (A_{\pm}(x, y) \nabla_{x,y} u_{\pm}) + \sum_{\pm} H_{\pm} (W \cdot \nabla_{x,y} u_{\pm} + V u_{\pm}) \quad (2.10)$$

in (2.8).

3 Three-region inequalities

Based on the Carleman estimate given in Theorem 2.1, we will derive three-region inequalities across the interface $y = 0$. Here we consider $u = H_+ u_+ + H_- u_-$ satisfying

$$\mathcal{L}(x, y, \partial)u = 0 \quad \text{in } \mathbb{R}^n,$$

where \mathcal{L} is given in (2.10) and

$$\|W\|_{L^\infty(\mathbb{R}^n)} + \|V\|_{L^\infty(\mathbb{R}^n)} \leq \lambda_0^{-1}.$$

Fix any $\delta \leq \delta_0$, where δ_0 is given in Theorem 2.1.

thm9.1

Theorem 3.1 Let u and $A_{\pm}(x, y)$ satisfy (2.1)-(2.6) with $h_0 = h_1 = 0$. Then there exist C and R , depending only on λ_0, M_0, n , such that if $0 < R_1, R_2 \leq R$, then

$$\int_{U_2} |u|^2 dx \leq (e^{\tau_0 R_2} + C R_1^{-4}) \left(\int_{U_1} |u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{U_3} |u|^2 dx dy \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \quad (3.1)$$

where τ_0 is the constant derived in Theorem 2.1,

$$\begin{aligned} U_1 &= \{z \geq -4R_2, \frac{R_1}{8a} < y < \frac{R_1}{a}\}, \\ U_2 &= \{-R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{8a}\}, \\ U_3 &= \{z \geq -4R_2, y < \frac{R_1}{a}\}, \end{aligned}$$

$a = \alpha_+/\delta$, and

$$z(x, y) = \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}. \quad (3.2)$$

Proof. To apply the estimate (2.8), u needs to satisfy the support condition. Also, we can choose α_+ and α_- in Theorem 2.1 such that $\alpha_+ > \alpha_-$. We can choose $r \leq r_0$ satisfying

$$r \leq \min \left\{ \frac{13\alpha_-}{8\beta}, \frac{2\delta}{19\alpha_- + 8\beta} \right\}. \quad (3.3)$$

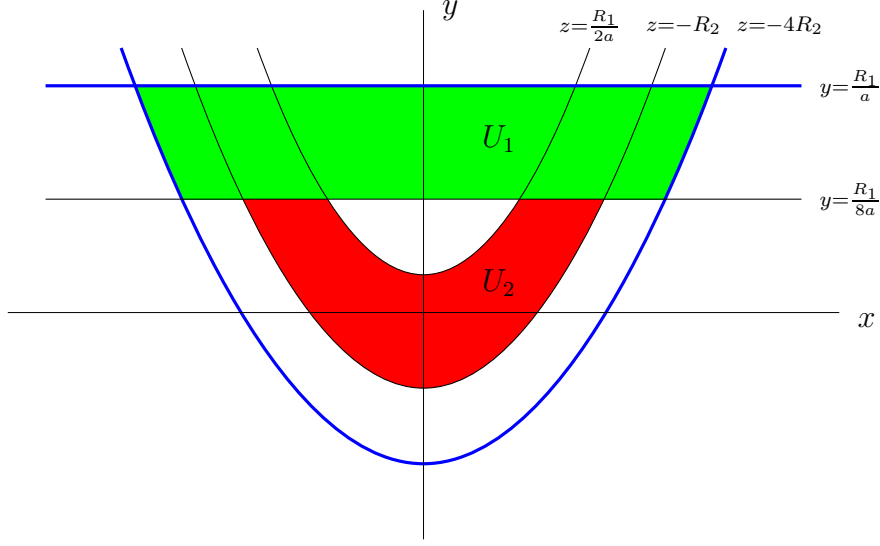


Figure 1: U_1 and U_2 are shown in green and red, respectively. U_3 is the region enclosed by blue boundaries. fig1

Note that the choices of δ, r also depend on λ_0, M_0, n . We then set

$$R = \frac{\alpha_- r}{16}.$$

It follows from (3.3) that

$$R \leq \frac{13\alpha_-^2}{128\beta}. \quad (3.4) \quad \text{bee}$$

Given $0 < R_1 < R_2 \leq R$. Let $\vartheta_1(t) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \vartheta_1(t) \leq 1$ and

$$\vartheta_1(t) = \begin{cases} 1, & t > -2R_2, \\ 0, & t \leq -3R_2. \end{cases}$$

Also, define $\vartheta_2(y) \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta_2(y) \leq 1$ and

$$\vartheta_2(y) = \begin{cases} 0, & y \geq \frac{R_1}{2a}, \\ 1, & y < \frac{R_1}{4a}. \end{cases}$$

Finally, we define $\vartheta(x, y) = \vartheta_1(z(x, y))\vartheta_2(y)$, where z is defined by (3.2).

We now check the support condition for ϑ . From its definition, we can see that $\text{supp } \vartheta$ is contained in

$$\begin{cases} z(x, y) = \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta} > -3R_2, \\ y < \frac{R_1}{2a}. \end{cases} \quad (3.5)$$

In view of the relation

$$\alpha_+ > \alpha_- \quad \text{and} \quad a = \frac{\alpha_+}{\delta},$$

we have that

$$\frac{R_1}{2a} < \frac{\delta}{2\alpha_-} \cdot R_1 < \frac{\delta}{\alpha_-} \cdot \frac{\alpha_- r}{16} < \delta r,$$

i.e., $y < \delta r \leq \delta r_0$. Next, we observe that

$$-3R_2 > -3R = -\frac{3\alpha_- r}{16} > \frac{\alpha_-}{\delta}(-\delta r) + \frac{\beta}{2\delta^2}(-\delta r)^2,$$

which gives $-\delta r < y$ due to (3.3). Consequently, we verify that $|y| < \delta r$. On the other hand, from the first condition of (3.5) and (3.3), we see that

$$\frac{|x|^2}{2\delta} < 3R_2 + \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} \leq \frac{3\alpha_- r}{16} + \frac{\alpha_-}{\delta} \cdot \delta r + \frac{\beta}{2\delta^2} \cdot \delta^2 r^2 \leq \frac{\delta}{8},$$

which gives $|x| < \delta/2$.

Since $h_0 = 0$, we have that

$$\vartheta(x, 0)u_+(x, 0) - \vartheta(x, 0)u_-(x, 0) = 0, \quad \forall x \in \mathbb{R}^{n-1}. \quad (3.6)$$

Applying (2.8) to ϑu and using (3.6) yields

$$\begin{aligned} & \sum_{\pm} \sum_{|k|=0}^2 \tau^{3-2|k|} \int_{\mathbf{R}_{\pm}^n} |D^k(\vartheta u_{\pm})|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy \\ & \leq C \sum_{\pm} \int_{\mathbf{R}_{\pm}^n} |\mathcal{L}(x, y, \partial)(\vartheta u_{\pm})|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy \\ & \quad + C\tau \int_{\mathbf{R}^{n-1}} |A_+(x, 0)\nabla_{x, y}(\vartheta u_+(x, 0)) \cdot \nu - A_-(x, 0)\nabla_{x, y}(\vartheta u_-(x, 0)) \cdot \nu|^2 e^{2\tau\phi_{\delta}(x, 0)} dx \\ & \quad + C[e^{\tau\phi_{\delta}(\cdot, 0)}(A_+(x, 0)\nabla_{x, y}(\vartheta u_+)(x, 0) \cdot \nu - A_-(x, 0)\nabla_{x, y}(\vartheta u_-)(x, 0) \cdot \nu)]_{1/2, \mathbf{R}^{n-1}}^2. \end{aligned} \quad (3.7)$$

We now observe that $\nabla_{x, y}\vartheta_1(z) = \vartheta'_1(z)\nabla_{x, y}z = \vartheta'_1(z)(-\frac{x}{\delta}, \frac{\alpha_-}{\delta} + \frac{\beta y}{\delta^2})$ and it is nonzero only when

$$-3R_2 < z < -2R_2.$$

Therefore, when $y = 0$, we have

$$2R_2 < \frac{|x|^2}{2\delta} < 3R_2.$$

Thus, we can see that

$$|\nabla_{x, y}\vartheta(x, 0)|^2 \leq CR_2^{-2} \left(\frac{6R_2}{\delta} + \frac{\alpha_-^2}{\delta^2} \right) \leq CR_2^{-2}. \quad (3.8)$$

By $h_0(x) = h_1(x) = 0$, (3.8), and the easy estimate of [5, Proposition 4.2], it is not hard to estimate

$$\begin{aligned}
& \tau \int_{\mathbf{R}^{n-1}} |A_+(x, 0) \nabla_{x,y}(\vartheta u_+(x, 0)) \cdot \nu - A_-(x, 0) \nabla_{x,y}(\vartheta u_-(x, 0)) \cdot \nu|^2 e^{2\tau\phi_\delta(x, 0)} dx \\
& + [e^{\tau\phi_\delta(\cdot, 0)} (A_+(x, 0) \nabla_{x,y}(\vartheta u_+)(x, 0) \cdot \nu - A_-(x, 0) \nabla_{x,y}(\vartheta u_-)(x, 0) \cdot \nu)]_{1/2, \mathbf{R}^{n-1}}^2 \\
& \leq C R_2^{-2} e^{-4\tau R_2} \left(\tau \int_{\{\sqrt{4\delta R_2} \leq |x| \leq \sqrt{6\delta R_2}\}} |u_+(x, 0)|^2 dx + [u_+(x, 0)]_{1/2, \{\sqrt{4\delta R_2} \leq |x| \leq \sqrt{6\delta R_2}\}}^2 \right) \\
& + C \tau^2 R_2^{-3} e^{-4\tau R_2} \int_{\{\sqrt{4\delta R_2} \leq |x| \leq \sqrt{6\delta R_2}\}} |u_+(x, 0)|^2 dx \\
& \leq C \tau^2 R_2^{-3} e^{-4\tau R_2} E,
\end{aligned} \tag{3.9}$$

where

$$E = \int_{\{\sqrt{4\delta R_2} \leq |x| \leq \sqrt{6\delta R_2}\}} |u_+(x, 0)|^2 dx + [u_+(x, 0)]_{1/2, \{\sqrt{4\delta R_2} \leq |x| \leq \sqrt{6\delta R_2}\}}^2.$$

Expanding $\mathcal{L}(x, y, \partial)(\vartheta u_\pm)$ and considering the set where $D^\vartheta \neq 0$, we can estimate

$$\begin{aligned}
& \sum_{\pm} \sum_{|k|=0}^1 \tau^{3-2|k|} \int_{\{-2R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{4a}\}} |D^k u_\pm|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy \\
& \leq C \sum_{\pm} \sum_{|k|=0}^1 R_2^{2(|k|-2)} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{2a}\}} |D^k u_\pm|^2 e^{2\tau\phi_{\delta, \pm}(x, y)} dx dy \\
& + C \sum_{|k|=0}^1 R_1^{2(|k|-2)} \int_{\{-3R_2 \leq z, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |D^k u_+|^2 e^{2\tau\phi_{\delta, +}(x, y)} dx dy \\
& + C \tau^2 R_2^{-3} e^{-4\tau R_2} E \\
& \leq C \sum_{\pm} \sum_{|k|=0}^1 R_2^{2(|k|-2)} e^{-4\tau R_2} e^{2\tau \frac{(\alpha_+ - \alpha_-)}{\delta} \frac{R_1}{4a}} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{4a}\}} |D^k u_\pm|^2 dx dy \\
& + \sum_{|k|=0}^1 R_1^{2(|k|-2)} e^{2\tau \frac{\alpha_+}{\delta} \frac{R_1}{2a}} e^{2\tau \frac{\beta}{2\delta^2} (\frac{R_1}{2a})^2} \int_{\{z \geq -3R_2, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |D^k u_+|^2 dx dy \\
& + C \tau^2 R_2^{-3} e^{-4\tau R_2} E.
\end{aligned} \tag{3.10}$$

Let us denote $U_1 = \{z \geq -4R_2, \frac{R_1}{8a} < y < \frac{R_1}{a}\}$, $U_2 = \{-R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{8a}\}$. From (3.10) and interior estimates (Caccioppoli's type inequality), we can derive that

$$\begin{aligned}
& \tau^3 e^{-2\tau R_2} \int_{U_2} |u|^2 dx dy \\
& \leq \tau^3 e^{-2\tau R_2} \int_{\{-R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{8a}\}} |u|^2 dx dy \\
& \leq \sum_{\pm} \tau^3 \int_{\{-2R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{4a}\}} |u_{\pm}|^2 e^{2\tau \phi_{\delta, \pm}(x, y)} dx dy \\
& \leq C \sum_{\pm} \sum_{|k|=0}^1 R_2^{2(|k|-2)} e^{-4\tau R_2} e^{2\tau \frac{(\alpha_+ - \alpha_-)}{\delta} \frac{R_1}{4a}} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{4a}\}} |D^k u_{\pm}|^2 dx dy \\
& \quad + \sum_{|k|=0}^1 R_1^{2(|k|-2)} e^{2\tau \frac{\alpha_+}{\delta} \frac{R_1}{2a}} e^{2\tau \frac{\beta}{2\delta^2} (\frac{R_1}{2a})^2} \int_{\{z \geq -3R_2, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |D^k u_+|^2 dx dy \tag{3.11} \\
& \quad + C\tau^2 R_2^{-3} e^{-4\tau R_2} E \\
& \leq C R_1^{-4} e^{-3\tau R_2} \int_{\{-4R_2 \leq z \leq -R_2, y < \frac{R_1}{a}\}} |u|^2 dx dy + C\tau^2 R_2^{-3} e^{-4\tau R_2} E \\
& \quad + C R_1^{-4} e^{(1 + \frac{\beta R_1}{4\alpha_-^2})\tau R_1} \int_{\{z \geq -4R_2, \frac{R_1}{8a} < y < \frac{R_1}{a}\}} |u|^2 dx dy \\
& \leq C R_1^{-4} \left(e^{2\tau R_1} \int_{U_1} |u|^2 dx dy + \tau^2 e^{-3\tau R_2} F \right),
\end{aligned}$$

where

$$F = \int_{\{z \geq -4R_2, y < \frac{R_1}{a}\}} |u|^2 dx dy$$

and we used the inequality $\frac{\beta R_1}{4\alpha_-^2} < 1$ due to (3.4).

Dividing $\tau^3 e^{-2\tau R_2}$ on both sides of (3.11) implies that

$$\int_{U_2} |u|^2 dx dy \leq C R_1^{-4} \left(e^{2\tau(R_1+R_2)} \int_{U_1} |u|^2 dx dy + e^{-\tau R_2} F \right). \tag{3.12}$$

Now, we consider two cases. If $\int_{U_1} |u|^2 dx dy \neq 0$ and

$$e^{2\tau_0(R_1+R_2)} \int_{U_1} |u|^2 dx dy < e^{-\tau_0 R_2} F,$$

then we can pick a $\tau > \tau_0$ such that

$$e^{2\tau(R_1+R_2)} \int_{U_1} |u|^2 dx dy = e^{-\tau R_2} F.$$

Using such τ , we obtain from (3.12) that

$$\begin{aligned} \int_{U_2} |u|^2 dx dy &\leq C R_1^{-4} e^{2\tau(R_1+R_2)} \int_{U_1} |u|^2 dx dy \\ &= C R_1^{-4} \left(\int_{U_1} |u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} (F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}. \end{aligned} \quad (3.13)$$

If $\int_{U_1} |u|^2 dx dy = 0$, then letting $\tau \rightarrow \infty$ in (3.12) we have $\int_{U_2} |u|^2 dx dy = 0$ as well. The three-regions inequality (3.1) obviously holds.

On the other hand, if

$$e^{-\tau_0 R_2} F \leq e^{2\tau_0(R_1+R_2)} \int_{U_1} |u|^2 dx dy,$$

then we have

$$\begin{aligned} \int_{U_2} |u|^2 dx &\leq (F)^{\frac{R_2}{2R_1+3R_2}} (F)^{\frac{2R_1+2R_2}{2R_1+3R_2}} \\ &\leq \exp(\tau_0 R_2) \left(\int_{U_1} |u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} (F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}. \end{aligned} \quad (3.14)$$

Putting together (3.13), (3.14), we arrive at

$$\int_{U_2} |u|^2 dx \leq (\exp(\tau_0 R_2) + C R_1^{-4}) \left(\int_{U_1} |u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} (F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}. \quad (3.15)$$

□

4 Size estimate

We will apply the three-region inequality (3.1) to estimate the size of embedded inclusion in this section. Here we denote Ω a bounded open set in \mathbb{R}^n with $C^{1,\alpha}$ boundary $\partial\Omega$ with constants s_0, L_0 , where $0 < \alpha \leq 1$. Assume that Σ is a C^2 hypersurface with constants r_0, K_0 satisfying

$$\text{dist}(\Sigma, \partial\Omega) \geq d_0 \quad (4.1)$$

for some $d_0 > 0$. We divide Ω into three sets, namely,

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-$$

where Ω_{\pm} are open subsets. Note that $\overline{\Omega_-} = \partial\Omega \cup \Sigma$ and $\partial\Omega_+ = \Sigma$. We also define

$$\Omega_h = \{x \in \Omega : \text{dist}(x, \partial\Omega) > h\}.$$

Definition 4.1 [$C^{1,\alpha}$ regularity] We say that Σ is C^2 with constants r_0, K_0 if for any $P \in \Sigma$ there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega_{\pm} \cap B(0, r_0) = \{(x, y) \in B(0, r_0) \subset \mathbb{R}^n : y \gtrless \psi(x)\},$$

where ψ is a C^2 function on $B_{r_0}(0)$ satisfying $\psi(0) = 0$ and

$$\|\psi\|_{C^2(B_{r_0}(0))} \leq K_0.$$

The definition of $C^{1,\alpha}$ boundary is similar. Note that $B(a, r)$ stands for the n -ball centered at a with radius $r > 0$. We remind the reader that $B_r(a)$ denotes the $(n-1)$ -ball centered at a with radius $r > 0$.

Assume that $A_{\pm} = \{a_{ij}^{\pm}(x, y)\}_{i,j=1}^n$ satisfy (2.3) and (2.4). Let us define $H_{\pm} = \chi_{\Omega_{\pm}}$, $A = H_+ A_+ + H_- A_-$, $u = H_+ u_+ + H_- u_-$. We now consider the conductivity equation

$$\operatorname{div}(A \nabla u) = 0 \quad \text{in } \Omega. \quad \text{a.e.} \quad (4.2)$$

It is not hard to check that u satisfies homogeneous transmission conditions (2.5), (2.6) (with $h_0 = h_1 = 0$), where in this case ν is the outer normal of Σ . For $\phi \in H^{1/2}(\partial\Omega)$, let u solve (4.2) and satisfy the boundary value $u = \phi$ on $\partial\Omega$.

Next we assume that D is a measurable subset of Ω . Suppose that \hat{A} is a symmetric $n \times n$ matrix with $L^\infty(\Omega)$ entries. In addition, we assume that there exist $\eta > 0, \zeta > 1$ such that

$$(1 + \eta)A \leq \hat{A} \leq \zeta A \quad \text{a.e. in } \Omega \quad \text{jump1} \quad (4.3)$$

or $\eta > 0, 0 < \zeta < 1$ such that

$$\zeta A \leq \hat{A} \leq (1 - \eta)A \quad \text{a.e. in } \Omega. \quad \text{jump2} \quad (4.4)$$

Now let $v = H_+ v_+ + H_- v_-$ be the solution of

$$\begin{cases} \operatorname{div}((A \chi_{\Omega \setminus \bar{D}} + \hat{A} \chi_D) \nabla v) = 0 & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega. \end{cases} \quad \text{byp2} \quad (4.5)$$

The inverse problem considered here is to estimate $|D|$ by the knowledge of $\{\phi, A \nabla v \cdot \nu|_{\partial\Omega}\}$. In this work we would like to consider the most interesting case where

$$\bar{D} \subseteq \bar{\Omega}_+. \quad \text{interior} \quad (4.6)$$

In practice, one could think of Ω_+ being an organ and D being a tumor. The aim is to estimate the size of D by measuring one pair of voltage and current on the surface of the body.

We denote W_0 and W the powers required to maintain the voltage ϕ on $\partial\Omega$ when the inclusion D is absent or present. It is easy to see that

$$W_0 = \int_{\partial\Omega} \phi A \nabla u \cdot \nu = \int_{\Omega} A \nabla u \cdot \nabla u$$

and

$$W = \int_{\partial\Omega} \phi(A_{\chi_{\Omega \setminus \bar{D}} + \hat{A}_{\chi_D}}) \nabla v \cdot \nu = \int_{\Omega} (A_{\chi_{\Omega \setminus \bar{D}} + \hat{A}_{\chi_D}}) \nabla v \cdot \nabla v.$$

The size of D will be estimate by the power gap $W - W_0$. To begin, we recall the following energy inequalities proved in [4].

Lemma 4.1 [4, Lemma 2.1] ^{energy} Assume that A satisfies the ellipticity condition (2.3). If either (4.3) or (4.4) holds, then

$$C_1 \int_D |\nabla u|^2 \leq |W_0 - W| \leq C_2 \int_D |\nabla u|^2, \quad \text{ineq1} \quad (4.7)$$

where C_1, C_2 are constants depending only on λ, η , and ζ .

The derivation of bounds on $|D|$ will be based on (4.7) and the following Lipschitz propagation of smallness for u .

Proposition 4.1 (Lipschitz propagation of smallness) ^{lippro} Let $u \in H^1(\Omega)$ be the solution of (4.2) with Dirichlet data ϕ . For any $B(x, \rho) \subset \Omega_+$, we have that

$$\int_{B(x, \rho)} |\nabla u|^2 \geq C \int_{\Omega} |\nabla u|^2, \quad \text{lp} \quad (4.8)$$

where C depends on $\Omega_{\pm}, d_0, \lambda_0, M_0, r_0, K_0, s_0, L_0, \alpha, \alpha', \rho$, and

$$\frac{\|\phi - \phi_0\|_{C^{1, \alpha'}(\partial\Omega)}}{\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}},$$

for $\phi_0 = |\partial\Omega|^{-1} \int_{\partial\Omega} \phi$. Here α' satisfies $0 < \alpha' < \frac{\alpha}{(\alpha+1)n}$.

Before proving Proposition 4.1, we need to adjust the three-region inequality (3.1) for the C^2 interface Σ . Let $0 \in \Sigma$ and the coordinate transform $(x', y') = T(x, y) = (x, y - \psi(x))$ for $x \in B_{s_0}(0)$. Denote $\tilde{U} = T(B(0, s_0))$ and $\tilde{\mathcal{A}}_{\pm} = \{\tilde{a}_{i,j}^{\pm}\}_{i,j=1}^n$ the coefficients of A_{\pm} in the new coordinates (x', y') . It is easy to see that $\tilde{\mathcal{A}}_{\pm}$ satisfies (2.3) and (2.4) with possible different constants $\tilde{\lambda}_0, \tilde{M}_0$, depending on λ_0, M_0, r_0, K_0 . Then there exist C and \tilde{R} , depending on $\tilde{\lambda}_0, \tilde{M}_0, n$, such that for

$$0 < R_1 < R_2 \leq \tilde{R} \quad \text{r1r2} \quad (4.9)$$

and U_1, U_2, U_3 defined as in Theorem 3.1, we have that $U_3 \subset \tilde{U}$ (so U_1, U_2 are contained in \tilde{U} as well) and (3.1) holds. Now let $\tilde{U}_j = T^{-1}(U_j)$, $j = 1, 2, 3$, then (3.1) becomes

$$\int_{\tilde{U}_2} |u|^2 dx dy \leq C \left(\int_{\tilde{U}_1} |u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{\tilde{U}_3} |u|^2 dx dy \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \quad \text{3r} \quad (4.10)$$

where C depends on $\lambda_0, M_0, r_0, K_0, n, R_1, R_2$. Furthermore, by Caccioppoli's inequality and generalized Poincaré's inequality (see (3.8) in [2]), we obtain from (4.10) that

$$\int_{\tilde{U}_2} |\nabla u|^2 dx dy \leq C \left(\int_{\tilde{U}_1} |\nabla u|^2 dx dy \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{\tilde{U}_3} |\nabla u|^2 dx dy \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}} \quad (4.11)^{3rd}$$

with a possibly different constant C .

Since A_+ (respectively A_-) is Lipschitz in Ω_+ (respectively Ω_-), the following three-sphere inequality is well-known. Let u_{\pm} be a solution to $\operatorname{div}(A_{\pm} \nabla u_{\pm}) = 0$ in Ω_{\pm} . Then for $B(x_0, \bar{r}) \subset \Omega_+$ (or $B(x_0, \bar{r}) \subset \Omega_-$) and $0 < r_1 < r_2 < r_3 < \bar{r}$, we have that

$$\int_{B(x_0, r_2)} |\nabla u_{\pm}|^2 dx dy \leq C \left(\int_{B(x_0, r_1)} |\nabla u_{\pm}|^2 dx dy \right)^{\theta} \left(\int_{B(x_0, r_3)} |\nabla u_{\pm}|^2 dx dy \right)^{1-\theta}, \quad (4.12)^{3s}$$

where $0 < \theta < 1$ and C depend on $\lambda_0, M_0, n, r_1/r_3, r_2/r_3$.

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. It suffices to study the case where ρ is small. Since $\Sigma \in C^2$, it satisfies both the uniform interior and exterior sphere properties, i.e., there exists $a_0 > 0$ such that for all $z \in \Sigma$, there exist balls $B \subset \Omega_+$ and $B' \subset \Omega_-$ of radius a_0 such that $\overline{B} \cap \Sigma = \overline{B'} \cap \Sigma = \{z\}$. Next let ν_z be the unit normal at $z \in \Sigma$ pointing into Ω_+ (inwards) and $L = \{z + t\nu_z \in \mathbb{R}^n : t \in [\rho_0, -3\rho_0]\}$. We then fix R_1, R_2 satisfying (4.9) and choose $\rho_0 > 0$ so that

$$S_z = \cup_{y \in L} B(y, \rho_0) \subset \tilde{U}_2.$$

Denote $\kappa = R_2/(2R_1 + 3R_2)$. Note that we move the construction of the three-region inequality from 0 to z .

Let $x \in \Omega_+$ and consider $B(x, \rho) \subset \Omega_+$, where $\rho \leq \min\{a_0, \rho_0\}$. For any $y \in \Omega_{2\rho}$, we discuss three cases.

(i) Let $y \in \Omega_{+, \rho}$, then by (4.12) and the chain of balls argument, we have that

$$\frac{\int_{B(y, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\theta^{N_1}}, \quad (4.13)^{e1}$$

where N_1 depends on Ω_+ and ρ .

(ii) Let $y \in \{\bar{\Omega}_+ : \operatorname{dist}(y, \Sigma) \leq \rho\} \cup \{\Omega_- : \operatorname{dist}(y, \Sigma) \leq 3\rho\}$, then $B(y, \rho) \subset S_z$ for some $z \in \Sigma$. Note that $\tilde{U}_1 \subset \Omega_{+, \rho}$ (taking ρ even smaller if necessary). We then apply (4.13) iteratively to estimate

$$\frac{\int_{\tilde{U}_1} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\theta^{N_1}}, \quad (4.14)^{e2}$$

where C depends on \tilde{U}_1 and ρ . Combining estimates (4.14) and (4.11) yields

$$\frac{\int_{B(y,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta^{N_1}}. \quad (4.15)$$

(iii) Finally, we consider the case where $y \in \Omega_- \cap \Omega_{2\rho}$ and $\text{dist}(y, \Sigma) > 3\rho$. We observe that if $y_* = z + (-3\rho)\nu_z$, then (4.15) implies

$$\frac{\int_{B(y_*,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta^{N_1}}. \quad (4.16)$$

Again using (4.12) and the chain of balls argument (starting with (4.16)), we obtain that

$$\frac{\int_{B(y,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta^{N_1} \theta^{N_2}}. \quad (4.17)$$

Putting together (4.13), (4.15), and (4.17) gives

$$\frac{\int_{B(y,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^s \quad (4.18)$$

for all $y \in \Omega_{2\rho}$, where $0 < s < 1$ and C depends on $\lambda_0, M_0, n, r_0, K_0, \rho, \Omega_{\pm}$.

In view of (4.18) and covering $\Omega_{3\rho}$ with balls of radius ρ , we have that

$$\frac{\int_{\Omega_{3\rho}} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^s. \quad (4.19)$$

Note that $u - \phi_0$ is the solution to (4.2) with Dirichlet boundary value $\phi - \phi_0$. By Corollary 1.3 in [13], we have that

$$\|\nabla u\|_{L^\infty(\Omega)}^2 = \|\nabla(u - \phi_0)\|_{L^\infty(\Omega)}^2 \leq C \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}^2$$

with $0 < \alpha' < \frac{\alpha}{(\alpha+1)n}$, which implies

$$\int_{\Omega \setminus \Omega_{3\rho}} |\nabla u|^2 \leq C |\Omega \setminus \Omega_{5\rho}| \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}^2 \leq C \rho \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}^2. \quad (4.20)$$

Here we have used $|\Omega \setminus \Omega_{5\rho}| \lesssim \rho$ proved in [3]. Using the Poincaré inequality, we have

$$\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}^2 \leq C \|u - \phi_0\|_{H^1(\Omega)}^2 \leq C \|\nabla u\|_{L^2(\Omega)}^2.$$

Combining this and (4.20), we see that if ρ is small enough depending on $\Omega_{\pm}, d_0, \lambda_0, M_0, r_0, K_0, s_0, L_0, \alpha, \alpha', \rho$, and $\|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)} / \|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}$, then

$$\frac{\|\nabla u\|_{L^2(\Omega_{3\rho})}^2}{\|\nabla u\|_{L^2(\Omega)}^2} \geq \frac{1}{2}.$$

The proposition follows from this and (4.19). \square

We now have enough tools to derive bounds on $|D|$.

sizeest1

Theorem 4.2 *Suppose that the assumptions of this section hold.*

(i) *If, moreover, there exists $h > 0$ such that*

$$|D_h| \geq \frac{1}{2}|D| \quad (\text{fatness condition}). \quad \text{fatness} \quad (4.21)$$

Then there exist constants $K_1, K_2 > 0$ depending only on Ω_\pm , d_0 , h , λ_0 , M_0 , r_0 , K_0 , s_0 , L_0 , α , α' , and $\|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}$, such that

$$K_1 \left| \frac{W_0 - W}{W_0} \right| \leq |D| \leq K_2 \left| \frac{W_0 - W}{W_0} \right|.$$

(ii) *For a general inclusion D contained strictly in Ω_+ , we assume that there exists $d_1 > 0$ such that*

$$\text{dist}(D, \partial\Omega_+) \geq d_1.$$

Then there exist constants K_1, K'_2 , $p > 1$, depending only on Ω_\pm , d_0 , d_1 , h , λ_0 , M_0 , r_0 , K_0 , s_0 , L_0 , α , α' , and $\|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}$, such that

$$K_1 \left| \frac{W_0 - W}{W_0} \right| \leq |D| \leq K'_2 \left| \frac{W_0 - W}{W_0} \right|^{\frac{1}{p}}. \quad \text{size101} \quad (4.22)$$

Proof. The proof follows closely the arguments in [4] and [17]. The lower bound can be obtained by basic estimates. Let $c = \frac{1}{|\Omega_{d/4}|} \int_{\Omega_{d/4}} u$. By the gradient estimate of [13, Theorem 1.1], the interior estimate of [9, Theorem 8.17] and the Poincaré inequality for the domain $\Omega_{d/4}$, we have

$$\|\nabla u\|_{L^\infty(\Omega_{d/2})} \leq C\|u - c\|_{L^\infty(\Omega_{d/3})} \leq C\|u - c\|_{L^2(\Omega_{d/4})} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

From this, the trivial estimate $\|\nabla u\|_{L^2(D)}^2 \leq C|D|\|\nabla u\|_{L^\infty(\Omega_{d/2})}^2$ and the second inequality of (4.7), the lower bound follows.

Next, we prove the upper bounds.

(i) Let $\rho = \frac{h}{4}$ and cover D_h with internally nonoverlapping closed squares $\{Q_k\}_{k=1}^J$ of side length 2ρ . It is clear that $Q_k \subset D$, hence

$$\begin{aligned} \int_D |\nabla u|^2 dx &\geq \int_{\cup_{k=1}^J Q_k} |\nabla u|^2 dx \geq \frac{|D_h|}{\rho^2} \min_k \int_{Q_k} |\nabla u|^2 dx \\ &\geq \frac{C|D|}{\rho^2} \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

Here we have used Proposition 4.1 and the fatness condition at the last inequality. The upper bound of $|D|$ follows from this and the first inequality of (4.7).

(ii) To prove the upper bound without the fatness condition, we need the fact that $|\nabla u|^2$ is an A_p weight which is an easy consequence of the doubling condition for ∇u . It turns out when D is strictly contained in Ω_+ where the coefficient A_+ is Lipschitz. The well-known theorem guarantees that $|\nabla u|^2$ is an A_p weight in Ω_+ (see [8] or [4]), i.e., for any $\bar{r} > 0$, there exists $B > 0$ and $p > 1$ such that

$$\left(\frac{1}{|B(a, r)|} \int_{B(a, r)} |\nabla u|^2 \right) \left(\frac{1}{|B(a, r)|} \int_{B(a, r)} |\nabla u|^{-\frac{2}{p-1}} \right)^{p-1} \leq B$$

for any ball $B(a, r) \subset \Omega_{+, \bar{r}}$, where B and p depends on various constants listed in Proposition 4.1. To derive the upper bound of (4.22), we choose $\bar{r} = d_1/2$ and follow exactly the same lines as in the proof of Theorem 2.2 [4].

□

Remark 4.3 We point out that part (i) of Theorem 4.2 still holds if the assumption (4.6) is replaced by

$$\text{dist}(D, \partial\Omega) \geq d_2 > 0.$$

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